HISTORICAL NOTE ON HEAVISIDE'S OPERATIONAL METHOD.

By N. W. McLachlan.

Operational methods in pure mathematics, *e.g.* the work of Boole, Graves and Murphy, existed many years before Oliver Heaviside introduced new ideas into mathematical physics. His first paper using such ideas was published in the *Proceedings of the Royal Society* in 1893.* Remarks therein indicate his unfamiliarity with some of the literature of the subject, owing to lack of reference facilities, for example, Liouville's work on fractional differentiation. Heaviside's main contributions to operational method seem to be: (*a*) the substitution $p^\nu \equiv (d/dt)^\nu$; (*b*) expansion in inverse powers of p and "algebrisation" using the formula $p^{-\nu} \doteqdot t^\nu/\Gamma(1+\nu)$; (*c*) incorporation of the initial conditions prior to solution; (*d*) the expansion theorem; and (*e*) expansion in ascending fractional powers of p to obtain an asymptotic formula. His main purpose was to solve transient problems in cable telegraphy and telephony, so he did not justify the analytical processes used. His method always yielded (in his hands!) the correct result, and he argued that this was in itself ample justification for the operational method. He wrote "The best proof is to go and do it".† Moreover, contemporary pure mathematicians looked upon his work with scepticism until such time as it was put on a sound foundation by Bromwich ‡ and Wagner § working independently. The boycotting of his work elicited some apposite remarks from Heaviside, as, for example: "Even Cambridge mathematicians deserve justice. . . . As regards their want of sympathy with less conventional men, it is not sympathy that is particularly wanted. . . . What one has a right to expect, however, is a fair field, and that the want of sympathy should be kept in a neutral state, so as not to lead to unnecessary obstruction."

"Orthodox mathematicians, when they cannot find the solution of a problem in a plain algebraical form, are apt to take refuge in a definite integral, and call it a solution. . . . But it may be just as hard, or harder, to interpret than the differential equation of the problem in question." ||

This psycho-polemical phase of the subject has passed away (we hope!), and Heaviside's operational calculus is now adopted (sometimes with considerable modification as will be shown later) as a standard technique in the solution of certain classes of problems, frequently those involving transient effects and wave motion. In the memorable paper where Bromwich ** established the validity of

* *Proc. Roy. Soc. A.* 52, 504 (1893).

† *E. M. Theory,* vol. 2, 34 (1922). ‡ *Proc. L.M.S.* (2), 15, 412 (1916).

§ *Archiv für Elektrotechnik* 4, Band 5, 159 (1916).

|| *E.M.T.* vol. 2, pp. 10, 11 (1922 edition). We disagree with Heaviside: see for example *Math. Gaz.* 22, 37, 1938, where a compact solution of a cable problem includes a definite integral which can be interpreted physically.

** *loc. cit.*

Heaviside's expansion theorem, he writes, " The investigations leading up to the formula (the expansion theorem) * are most instructive and will repay careful study : but I am not sure that they have been sufficiently appreciated in the past. It is almost certain that few readers have fully grasped the complete and general character of the solution : and it is for this reason that I wish to call attention to it here."

Some forty years after publication of the expansion theorem, one of the leading authorities of the Cambridge school of mathematicians wrote : " We should now place the operational calculus with Poincaré's discovery of automorphic functions and Ricci's discovery of the tensor calculus, as the three most important mathematical advances of the last quarter of the nineteenth century. Applications, extensions, and justifications of it constitute a considerable part of the mathematical activity of to-day." †

Bromwich ‡ established the validity of Heaviside's method of operators—at least in so far as it involved the evaluation of residues at poles—by complex integration. Amongst other things, he pointed out that when $f(t)$ is zero outside the range $0 < t < \tau$, if

$$\xi(p) = \int_0^\tau e^{-pt} f(t)\,dt, \qquad \text{.............................(1)}$$

then

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\,\xi(p)\,dp, \qquad \text{..................(2)}$$

where $\xi(p)$ is to be regarded as a convenient transform for $f(t)$.

Suppose we solve the equation

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = f_0 \sin t, \qquad \text{........................(3)}$$

subject to the conditions $y' = y = 0$ at $t = 0$, by the Bromwich method. Then we assume that §

$$y = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\,\chi(p)\,dp,$$

which on substitution in the left-hand side of (3) yields

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\,\chi(p)\{p^2 + ap + b\}\,dp. \qquad \text{..................(4)}$$

Since $\sin t$ continues indefinitely, $\tau = \infty$ in (1), so

$$\xi(p) = f_0 \int_0^\infty e^{-pt} \sin t\,dt = f_0/(p^2+1), \qquad \text{...............(5)}$$

* *Electrical Papers*, vol. 2, p. 259.

† E. T. Whittaker, " Oliver Heaviside ", *Bull. Math. Soc. Calcutta*, 20, 199 (1928-29).

‡ *loc. cit.*

§ All the singularities of the integrand lie to the left of the contour. If they are all poles, $c \pm i\infty$ may be replaced by a circle enclosing them.

and by (2)

$$f_0 \sin t = \frac{f_0}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\, dp/(p^2+1). \qquad \text{......................(6)}$$

Hence by (4) and (6)

$$\chi(p) = f_0/(p^2+ap+b)(p^2+1), \qquad \text{......................(7)}$$

so
$$y = \frac{f_0}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\, dp/(p^2+ap+b)(p^2+1), \qquad \text{......(8)}$$

which is evaluated in the usual way by the calculus of residues.

The origin of formulae (1) and (2) is sometimes misquoted by writers on the subject of operators, so we give Bromwich's remarks * thereon *verbatim* : " The formulae may be deduced, as a matter of simple transformation, from the ordinary form of Fourier's theorem. Formulae substantially equivalent were obtained in this way by Riemann in his paper on the distribution of primes (*Ges. Werke*, p. 140) ; and the actual formulae were deduced similarly by Macdonald (*Proc. L.M.S.* 35, 428 ; 1902). From the point of view of the theory of functions of a complex variable, more complete discussions have been given by Pincherle and Mellin ; see for instance a paper by the latter in *Math. Annalen*, Band 68, p. 305, where references to earlier investigations will be found."

Heaviside did not seem to be favourably impressed by the method of contour integration, as will be realised on reading the following excerpt from a letter † he wrote to Bromwich on 7th April, 1919, from Torquay, where he lived till his demise in 1925 :

" DEAR DR. BROMWICH,

Yours 5:4:19, Caesar and Pompey : especially Pompey. . . . What a time it takes! I rejoice to know that you have seen the simplicity and advantages of my way. Only a war could have done it. Did you see in the paper that I was the cause of the war, and that I brought it on to wake up England and especially Cambridge? Now let the wooden headed rigorists go hang, and stick to differential operators and leave out the rigorous footnotes. It is easy enough if you don't stop to worry. As I said and Lord R. ‡ repeated, logic is the very last thing. I never could stomach your complex integral method, though I understand your affection for it. . . . I don't care much about the expansion theorem myself now, it is so tame. . . ."

After this brief résumé of Bromwich's work, we turn to Wagner, § who published a paper almost contemporaneous with that of the former writer. He commenced with the integral

$$f = \frac{f_0}{2\pi i}\int_{-i\infty}^{i\infty} e^{pt}\frac{dp}{p} = \begin{cases} 0, & t < 0 \\ f_0, & t > 0, \end{cases} \qquad \text{......................(9)}$$

* *loc. cit.* p. 412.

† The original was given to A. T. Starr when at Cambridge under Bromwich. The author is indebted to the former for a lantern slide of the original script.

‡ Presumably the late Lord Rayleigh. § *loc. cit.*

taking the imaginary axis indented on the right at the origin as contour. Integral (9) was used to define Heaviside's unit function. Wagner went on to state that Heaviside's first expansion theorem was equivalent to evaluation of an integral of the form

$$\int_{-i\infty}^{i\infty} e^{pt}\,\phi(p)\,\frac{dp}{p}, \quad \text{..............................(10)}$$

by finding the residues at the poles of the integrand. No formal proof was given, however, but a number of examples from engineering practice were treated in a way comparable with that of Bromwich.

The subject of branch points was not mentioned in either of these papers. In fact it was not until some years later that the topics of branch points, equivalent contours (*e.g.* $-\infty$, $(0+)$, $-\infty$) and the asymptotic expansions used by Heaviside were considered with respect to the complex variable.

The contour used by Bromwich, namely, $c-i\infty$ to $c+i\infty$ is more general than that of Wagner, and it is now employed for the interpretation of an operational expression $\phi(p)$, as in the following integral:

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\,\phi(z)\,\frac{dz}{z}, \quad \text{..................(11)}$$

all singularities of the integrand being to the left of the contour. In its relation to operational procedure, this formula can be considered to be based upon (2) and (10). Accordingly it may be regarded as the Bromwich-Wagner integral. In modern practical applications it is useful for interpreting operational forms $\phi(p)$, where

$$\phi(p)=p\int_0^\infty e^{-pt} f(t)\,dt. \quad \text{........................(12)}$$

When certain conditions* are fulfilled, (11) and (12) constitute a particular case of the Mellin inversion theorem.† $\phi(p)/p$ is the Laplace transform of $f(t)$.

The contour $c\pm i\infty$, where $c>0$ is not necessarily constant, may for present purposes be regarded as the Bromwich contour. Provided the integrals along the arcs in the second and third quadrants, joining the extremities of the contours, vanish as their radii $\to\infty$, any contour starting in the second quadrant, crossing the real axis at $z=c_1>0$ and terminating in the third quadrant, is equivalent to $c\pm i\infty$. In certain cases the equivalent contour $-\infty$, $(0+)$, $-\infty$ is very useful, *e.g.* asymptotic expansions in cable problems.

Bromwich followed up his first paper in 1916 by a number of others dealing with various problems in mathematical physics. These contributions to the subject appeared from time to time during the years 1919-30.

* See McLachlan, *Phil. Mag.* 25, 261, 1938.

† Titchmarsh, *Introduction to Theory of Fourier Integrals* (1937).

Heaviside's procedure and the Mellin inversion theorem.

Those who have used Heaviside's operational procedure are aware of the possibility of pitfalls (Bromwich called them slippery places). Books have been written for technicians by technicians in which this procedure is crystallised, so to speak, into rules, and in a number of practical applications they are easy to apply. But the blind usage of a technique (which is never justified by formal processes and logical argument in the said books) invariably stimulates a psychological reaction, and the technician often finds himself in a dilemma. A number of engineers have informed me that one can never be certain of the Heaviside method, but it seems to be satisfactory provided the result can be checked in some other way! There are, however, problems whose solution by expansion in series and "algebrisation" in the Heaviside manner are unsatisfactory, the form of solution throwing little or no light upon the physics of the situation.*

Attempts have been made hitherto to justify the Heaviside procedure formally, without recourse to complex integration, and to show that if $p \equiv d/dt$, then p^{ν}, where ν is real, can be interpreted in terms of the gamma function. But such theses do not carry conviction.

By using the Mellin inversion theorem, as in the example given above, the operational solution of a problem is found as a function of p, *i.e.* $\phi(p)$, the former being a mere number whose real part > 0. There is no question of p being identified with d/dt. Having got the operational solution, it is interpreted in terms of t by aid of integral (11). It happens, however, that the same operational solution is obtained by writing p for d/dt, and the same function of t appears when the operational solution is algebrised in the Heaviside manner. Interpretation by the contour integral frequently gives the result in the form of a tabulated function, whereas algebrising may yield an unrecognisable infinite series.†

We may say that Heaviside's operational procedure "happened", as do many great inventions and events in life. They are none the worse for this. If experimental results could always be deduced *a priori*, research would be unnecessary, and this would deprive the scientist not only of his joy in life, but of his livelihood! The position respecting operational procedure may be summarised by saying that "something 'happened' to Heaviside which made Bromwich and Wagner 'function'," to the great advantage of mathematicians and technicians. With the march of time, it is necessary to replace old methods by new in every department of scientific activity, wherever this is feasible. Moreover, the time is now ripe to remember *what*

* See McLachlan, *Math. Gaz. loc. cit.*

† As an example of this we can take the case of $1/(1+ap^{\frac{1}{2}})$. The contour integral method yields $1 - e^{t/a^2}(1 - \operatorname{erf}\sqrt{t/a^2})$, but the series for this would not be easily recognised. This occurs in a problem on elasto-viscosity of materials like flour-dough or gelatine solution.

the illustrious Heaviside did, but to forget *how* he did it. So far as our present knowledge goes, the best procedure seems to be the application of the Mellin inversion theorem, as in the example given above.* When this is used, " p " is merely an inversion or transform parameter. Accordingly the nomenclature " operator " and " operational " is inaccurate. But in this matter we have to contend with usage extending over half a century, and an attempt to rechristen using, say, " Parametric " or " Transform " Calculus, would doubtless be greeted with ironical cheers. In any case a change of nomenclature would be unwise before being agreed upon at an International Conference. At some future date, when the Mellin theorem holds the operational field, the " young idea " will wonder how the word " operational " came into being to describe something entirely different. Some of the present generation are at a loss to explain why Thomson and Tait (T and T') perpetrated the ill-founded title " Spherical harmonics ". Musicians and acousticians should be sole concessionaires of the word *harmonics* !

Finally, we shall compare operational procedure with a time-honoured engineering process. Suppose we have a large ingot of steel and want to make an elaborate framework therefrom. By using hammers, chisels, hacksaws, drills and other engineering appliances, the ingot can be shaped ultimately into conformity with our design. This may be regarded as the sledge-hammer or battle-axe method. If the ingot is melted and run into a sand mould (made by aid of a wooden pattern of the framework), the required result is obtained very quickly and much more simply than by the sledge-hammer way. We may conceive the moulding procedure to be akin to the operational method in mathematics, the sledge-hammer method comparing with analysis prior to the operational era.

Transformation is made from the solid in a certain form to the liquid state, which after being given a predetermined shape is allowed to regain the solid state. The dependent variable y in a differential equation is expressed in terms of a parameter p, the resulting expression is interpreted, and the dependent variable y is obtained explicitly in terms of the independent variable t.

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1189. To offer the reader that sentence (from an Act of Parliament) . . . would be as little reasonable as the revenge offered to a village schoolmaster who, having complained that the whole of his little treatise on the Differential Calculus was printed bodily in one of the earlier editions of the *Encyclopaedia Britannica,* was told that he was welcome, in his turn, to incorporate the *Encyclopaedia Britannica* in the next edition of his little treatise.—J. Hill Burton, *The Book Hunter*, p. 144. [Per Mr. J. B. Bretherton.]

1190. Between these two parallel lines a gulf still opens. It is far narrower than it was ; but still as profound and still unbridged.—Julian Huxley, *Essays in Popular Science* (Pelican Books), p. 138. [Per Dr. C. Fox.]

* The most powerful method for solving technical problems of various kinds is that of integral equations. The Mellin theorem provides a ready method of solution for a certain class of problem.